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Original Article

# Semigroup Arrising From Trellises

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***Abstract***

*This study explores semigroups, which are algebraic structures with an associative binary operation, that arise from trellis structures. The research investigates the formation and properties of these semigroups. It is proven that the set of all functions from a trellis L to itself, denoted T(L), also forms a trellis. The concept of an order-preserving map in T(L) is discussed. The study demonstrates that the sets of all join homomorphisms on L (J(L)), all meet homomorphisms on L (M(L)), and all endomorphisms on L (E(L)) are submonoids of T(L). Additionally, it is observed that the set of all Automorphisms of L (Aut(L)) forms a group with respect to function composition. The study further discusses the conditions under which monoids like T(L), J(L), M(L), E(L), and Aut(L) are isomorphic to their corresponding monoids arising from another trellis L'.* **2020 Mathematical Sciences Classification:** 06D75.

**Keywords and Phrases:** semigroups, homomorphisms, endomorphisms, automorphisms.

## Introduction

The notion of a trellis was introduced by H.L. Skala [5] and Fried [1] as natural generalization of a lattice. This paper investigates semigroups, which are algebraic structures with an associative binary operation that arise from trellis structures. We prove the set all functions from trellis L into itself is also a trellis. The concept of order preserving map in T(L) is discussed. We prove that the sets J(L) (the set of all join homomorphisms on L), M(L) (the set of all meet homomorphisms on L) and E(L) (the set of all endomorphisms on L) are sub-monoids of T(L).

This paper explores how these semigroups are formed and their properties. It is observed that the set of all Automorphisms of L (Aut(L)) is group with respect to function composition.

## Semigroups of function

In this paper we focus our attention on the class of functions defined on a trellis which forms a semigroup under composition.

### Trellis structure on set of functions

Let, $(L, \vee, \wedge)$ be a trellis. Let $T(L)$ be the set of function that maps $L$ into itself.

For $f, g \in T(L)$ define $(f \sqcup g)(x) = f(x) \vee g(x)$ and $(f \sqcap g)(x) = f(x) \wedge g(x)$

Then $f \sqcup g$ and $f \sqcap g$ are elements of $(L)$. Moreover we have the following.

**Theorem 2.1.1.** *If $(L, \vee, \wedge)$ is a trellis, then $(T(L), \sqcup, \sqcap)$ is a trellis.*

***Proof.*** *i) Let $f \in T(L)$. Then for all $x \in L$,*

$(f \sqcup f)(x) = f(x) \vee f(x) = f(x)$

and $(f \sqcap f)(x) = f(x) \wedge f(x) = f(x)$

So $f \sqcup f = f$ and $f \sqcap f = f$.

*ii)* For $f, g \in T(L)$ and any $x \in L$ we have,

$(f \sqcup g)(x) = f(x) \vee g(x) = g(x) \vee f(x) = (g \sqcup f)(x)$ and $(f \sqcap g)(x) = f(x) \wedge g(x) = g(x) \wedge f(x) = (g \sqcap f)(x)$ Therefore $f \sqcup g = g \sqcup f$ and $f \sqcap g = g \sqcap f$.

*iii) For $f, g, h \in T(L)$ and any $x \in L$ we have,*

$(f \sqcup (g \sqcap f))(x) = f(x) \vee (g \sqcap f)(x) = f(x) \vee (g(x) \wedge f(x)) = f(x)$

and $(f \sqcap (g \sqcup f))(x) = f(x) \wedge (g \sqcup f)(x) = f(x) \wedge (g(x) \vee f(x)) = f(x)$ This proves that $f \sqcup (g \sqcap f) = f$ and $f \sqcap (g \sqcup f) = f$.



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$iv)$ For $f, g, h \in T(L)$ and any $x \in L$ we have,
$(f \sqcup ((f \sqcap g) \sqcup (f \sqcap h)))(x) = f(x) \vee ((f \sqcap g) \sqcup (f \sqcap h))(x)$
$= f(x) \vee ((f(x) \wedge g(x)) \vee f(x) \wedge h(x)))$
$= f(x)$
and $(f \sqcap ((f \sqcup g) \sqcap (f \sqcup h))(x) = f(x) \wedge ((f \sqcup g) \sqcap (f \sqcup h))(x)$
$= f(x) \wedge ((f(x) \vee g(x)) \wedge (f(x) \vee h(x)))$
$= f(x)$ From part $(i)$to $(iv)$ we conclude that $(T(L), \sqcup, \sqcap)$ is a trellis.

***Definitions***. An element $f \in T(L)$ is said to be a join (meet) homomorphism if $f(x \vee y) = f(x) \vee f(y)$ $(f(x \wedge y) = (x) \wedge (y))$ for all $x, y \in L$. The set of all join homomorphisms on $L$ is denoted by $J$ (L) and that of meet homomorphisms on L is denoted by $(L)$. An element of $T(L)$ which is both a join and meet homomorphism is known as an endomorphism of L and the set of all endomorphisms of L is denoted by E$(L)$. Clearly, E$(L)$ = $J(L) \cap M(L)$. If, in addition, f is one-to-one and onto, then f is an automorphism. The set of all automorphisms of $L$ is denoted by $A$ut$(L)$.

**Endomorphism semigroup of a trellis**

Let $(L, \vee, \wedge)$ be a trellis. Then $T(L)$ is a monoid under function composition, where the identity map $I$ on $L$ is the identity element in $T(L)$. A map $f : L \to L$ is called an order preserving map if for all $a, b \in L, a \trianglelefteq b$ implies $f(a) \trianglelefteq f(b)$. The set of all order preserving maps in $T(L)$ is denoted by $O(L)$.

**Proposition 2.3.1.** *If $(L,\vee,\wedge)$ is a trellis, then $O(L)$is submonoid of $T(L)$.*

***Proof***. Let $f, g \in O(L)$. Then for $a, b \in L$ we have,
$a \trianglelefteq b \Rightarrow g(a) \trianglelefteq g(b)$
$\Rightarrow f\, g(a) \trianglelefteq f\, g(b)$
$\Rightarrow (f \circ g)(a) \trianglelefteq (f \circ g)(b)$
Therefore $f \circ g \in O(L)$
It is clear that the Identity map $I$ on $L$ is an order preserving map i.e.$I \in O(L)$. Therefore $O(L)$ is a submonoid of $T(L)$.
Now onwards we write $fg$ for $f \circ g$.

**Theorem 2.3.2.** *Let $(L, \vee, \wedge)$ be a trellis Then J(L), M(L) and E(L) are submonoids of T(L). Also E(L) is submonoid of both $J(L)$ and M(L).*

***Proof.*** Let $f, g \in (L)$ then for all $x, y \in L$ we have
$(fg)(x \vee y) = f(g(x \vee y))$
$= f(g(x) \vee g(y))$
$= f(g(x)) \vee f(g(y))$
$= (fg)(x) \vee (fg)(y)$
Thus $fg \in$ J$(L)$. Since composition is associative and identity map on $L$ is in $J(L)$, we get $J(L)$ is a submonoid of $T(L)$. Similarly, we can prove that M$(L)$ and E$(L)$ are submonoids of $T(L)$. Since E$(L) = J(L) \cap$ M$(L)$ we have $E(L)$ is a submonoid of both J(L) and $M(L)$.

**Corollary 2.3.3.** *Let (L, ∨, ∧) be a trellis then Aut(L) is a group with respect to function composition.*

***Proof.*** Let $f, g \in A$ut$(L)$, then by Theorem 2.3.2, $fg \in E(L)$. Since composition of two bijections is a bijection, we have $fg \in A$ut$(L)$. The associative property of function composition in $A$ut$(L)$ follows trivially. As the identity map $I$ on $L$ is a bijective trellis homomorphism, we get $I \in A$ut$(L)$. Let $f \in A$ut$(L)$ and $x, y \in L$. Then there exist $x', y' \in L$ such that $f(x') = x$ and $f(y') = y$. Then by definition of $f^{-1}$, we have
$f^{-1}(x) = x'$ and $f^{-1}(y) = y'$. Now,
$f^{-1}(x \vee y) = f^{-1}(f(x') \vee f(y'))$
$= f^{-1}(f(x' \vee y'))$
$= (x' \vee y')$
$= f^{-1}(x) \vee f^{-1}(y)$
Therefore $f^{-1}(x \vee y) = f^{-1}(x) \vee f^{-1}(y) \forall x, y \in L$
Similarly, we can prove
$f^{-1}(x \wedge y) = f^{-1}(x) \wedge f^{-1}(y) \forall x, y \in L.$
Hence $f^{-1} \in$ Aut$(L)$ and $ff^{-1} = I = f^{-1}f$. This proves that $A$ut$(L)$ is a group.
In the following theorem we discuss the conditions under which the monoids $T(L)$,
J$(L)$, M$(L)$, E$(L)$ and $Aut(L)$ arising from a trellis L are isomorphic to the corresponding monoids arising from another trellis $L'$.

**Theorem 2.3.4.** *Let (L, ∨, ∧) and (L', ∨', ∧') be two trellises.*

*(i) If there is a one-one map from L onto L' then monoids T(L) and T(L') are isomorphic.*

*(ii) If there is join isomorphism from L onto L' then J(L) and J(L') are isomorphic.*

*(iii) If there is meet isomorphism from L onto L' then M(L) and M(L') are isomorphic.*

*(iv) If there is an isomorphism from L onto L' then E(L) and E(L') are isomorphic.*

*(v) If there is isomorphism from L onto L' then Aut(L) and Aut(L') are isomorphic.*

***Proof.*** (i). Let $\theta : L \rightarrow L'$ be a one-one and onto function. If $f \in T(L)$ and $h \in T(L')$, then $\theta f \theta^{-1} \in T(L')$ and $\theta^{-1} h \theta \in T(L)$.

Define $\phi : T(L) \longrightarrow T(L')$ by, $\phi(f) = \theta f \theta^{-1}$, for all $f \in T(L)$. Then $\phi$ is well-defined. For $f, g \in T(L)$ and $x \in L'$, using the fact that $\theta$ is one - one we have

$\phi(f) = \phi(g) \Rightarrow \theta f \theta^{-1} = \theta g \theta^{-1}$

$\Rightarrow (\theta f \theta^{-1})(x) = (\theta g \theta^{-1})(x)$

$\Rightarrow \theta(f(\theta^{-1}(x))) = \theta(g(\theta^{-1}(x)))$

$\Rightarrow f(\theta^{-1}(x)) = g(\theta^{-1}(x))$

Now for any $a \in L$, we can write $a = \theta^{-1}(\theta(a))$

Therefore $f(a) = f(\theta^{-1}((a)) = g(\theta^{-1}((a)) = g(a)$. Hence $f(a) = g(a)\ \forall a \in L$, which proves that $f = g$.

Thus $\phi(f) = \phi(g)$ implies $f = g$, which means $\phi$ is one-one.

Let $F \in T(L')$, then $\theta^{-1} F \theta \in (L)$ and $\phi(\theta^{-1} F \theta) = (\theta^{-1} F \theta)\theta^{-1} = F$. Therefore $\phi$ is onto

Let $f, g \in T(L)$. Then

$\phi(fg) = \theta(fg)\theta^{-1}$

$= \theta(f\theta^{-1}\theta g)\theta^{-1}$

$= (\theta f \theta^{-1})(\theta g \theta^{-1})$

$= \phi(f).\phi(g)$

Thus $\phi$ is a homomorphism and hence $\phi : T(L) \rightarrow T(L')$ is an isomorphism.

(ii). Let $\theta : L \rightarrow L'$ be a join isomorphism. Then it is easy to verify that $\theta^{-1} : L' \rightarrow L$ is also a join homomorphism. For $f \in J(L)$ we have

$(\theta f \theta^{-1})(x \vee' y) = \theta f(\theta^{-1}(x \vee' y))$

$= \theta f(\theta^{-1}(x) \vee \theta f(\theta^{-1}(y))$

$= \theta(f\theta^{-1}(x) \vee \theta(f\theta^{-1}(y))$

$= (\theta f \theta^{-1})(x) \vee' (\theta f \theta^{-1})(y)$

Thus $\theta f \theta^{-1} \in J(L')$. Similarly, it can be proved that if $h \in J(L')$ Then $\theta^{-1} h \theta \in J(L)$.

Define $\phi : J(L) \rightarrow J(L')$ by

$\phi(f) = \theta f \theta^{-1}, for\ all\ f \in J(L)$.

Then $\phi$ is well-defined and as proved in part (i) we can prove $\phi$ is an isomorphism between $J(L)$ and $J(L')$

(iii) If $\theta : L \rightarrow L'$ is a meet homomorphism, then $\theta f \theta^{-1} \in M(L')$ for all $f \in M(L)$ and $\theta^{-1} h \theta \in M(L)\ for\ all\ h \in M(L')$. Remaining part of the proof is similar to that of (i).

(iv) If $\theta : L \rightarrow L'$ is an isomorphism from $L\ to\ L'$, then $\theta$ is a meet and join homomorphism. Using part (ii) and (iii) and the fact that $E(L) = J(L) \cap M(L)$ it follows that

$\phi : E(L) \rightarrow E(L')$ defined by $\phi(f) = \theta f \theta^{-1}$, for all $f \in E(L)$ is a isomorphism from $E(L)\ to\ E(L')$.

(v) If $\theta : L \rightarrow L'$ is an isomorphism, then $\theta f \theta^{-1} \in Aut(L)$, for all $f \in Aut(L)$.

Define $\phi : Aut(L) \rightarrow Aut(L')$ by $\phi(f) = \theta f \theta^{-1}\ for\ all\ f \in Aut(L)$. Then, by (iv), $\phi$ is a group isomorphism.

**Conclusion**

The paper investigates semigroups of functions defined on a trellis and proves that the set of all functions from a trellis into itself is also a trellis. It discusses the concept of order-preserving maps and proves that the sets of join homomorphisms, meet homomorphisms, and endomorphisms are submonoids of the set of all functions from a trellis into itself.

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